Ext and Universal coefficient theorem

1. A : R-module (R: commutative ring with 1)
Let

$$\cdots \longrightarrow C_p \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\epsilon} A \xrightarrow{\partial} 0$$

be a free resolution of A.
Define $\mathrm{Ext}^p(A, G) := H^p(\mathcal{C}; G)$, the homology of

$$0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots$$

where $C^i = \mathrm{Hom}_R(C_i, G)$.
<u>Check</u>: This is well-defined, i.e., independent of choice of a free resolution:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial} & C_1 & \xrightarrow{\partial} & C_0 & \xrightarrow{\epsilon} & A & \longrightarrow & 0 \\
 & & {\scriptstyle i_1}\downarrow\uparrow{\scriptstyle j_1} & & {\scriptstyle i_0}\downarrow\uparrow{\scriptstyle j_0} & & \|{\scriptstyle id} & & \\
\cdots & \xrightarrow{\partial} & C'_1 & \xrightarrow{\partial} & C'_0 & \xrightarrow{\epsilon} & A & \longrightarrow & 0
\end{array}$$

$id : A \to A$ has a lifting and liftings are chain homotopic by comparison theorem. Therefore

$$j \circ i \simeq id, j \circ i \simeq id.$$

It follows that i is a chain homotopy equivalence and induces a cochain homotopy equivalence and hence an isomorphism

$$i^* : H^p(\mathcal{C}'; G) \xrightarrow{\cong} H^p(\mathcal{C}; G).$$

<u>Note</u>: Any such liftings i are all chain homotopic and hence induce a same homomorphism i^* and this is induced from $id : A \to A$ i.e., there exists a canonical isomorphism between $H^p(\mathcal{C}'; G)$ and $H^p(\mathcal{C}; G)$.

Properties
(1) $\mathrm{Ext}^0(A, G) = H^0(\mathcal{C}; G) = \mathrm{Hom}(A, G)$

$$0 \longrightarrow \mathrm{Hom}_R(A, G) \xrightarrow{\widetilde{\epsilon}} \mathrm{Hom}_R(C_0, G) \xrightarrow{\delta = \widetilde{\partial}} \mathrm{Hom}_R(C_1, G) \longrightarrow \cdots$$

is exact at $\mathrm{Hom}_R(C_0, G)$. So $\mathrm{Ext}^0(A, G) = ker\delta = im\widetilde{\epsilon} \cong \mathrm{Hom}(A, G)$.

(2) $\mathrm{Ext}^p(F, G) = 0$, if $p \geq 1$ and F is free.
A free resolution of F is

$$\cdots \longrightarrow 0 \longrightarrow F \xrightarrow{id} F \longrightarrow 0$$

So if $p \geq 1$, $C_p = 0$ and

$$\mathrm{Ext}^p(F, G) = H^p(\mathcal{C}; G) = 0.$$

(3) Similarly $\mathrm{Ext}^p(A, G) = 0$ if $p \geq 2$ and A is an abelian group or a R-module with PID R.
A free resolution of A is

$$\begin{array}{ccccccccc} \cdots \longrightarrow 0 \longrightarrow & R & \longrightarrow & F & \xrightarrow{\epsilon} & A & \longrightarrow 0. \\ & \| & & \| & & & \\ & ker\epsilon & & free & & & \end{array}$$

So if $p \geq 2$, $C_p = 0$ and

$$\mathrm{Ext}^p(F, G) = H^p(\mathcal{C}; G) = 0.$$

In this case we simply denote $\mathrm{Ext}(A, G)$ for $\mathrm{Ext}^1(A, G)$.

(4) $\mathrm{Ext}^p(-, G)$ is a contravariant functor.
For $\gamma : A \to A'$, by comparison theorem there exist liftings γ_0, γ_1, $\cdots$ such that the following diagram commutes.

$$\begin{array}{ccccccc} \cdots \xrightarrow{\partial} & C_1 & \xrightarrow{\partial} & C_0 & \xrightarrow{\epsilon} & A & \longrightarrow 0 \\ & \Big\downarrow{\scriptstyle \gamma_1} & & \Big\downarrow{\scriptstyle \gamma_0} & & \Big\downarrow{\scriptstyle \gamma} & \\ \cdots \xrightarrow{\partial} & C_1' & \xrightarrow{\partial} & C_0' & \xrightarrow{\epsilon} & A' & \longrightarrow 0 \end{array}$$

Since such liftings are unique up to chain homotopy, $\gamma_p^* : \mathrm{Ext}^p(A', G) = H^p(\mathcal{C}'; G) \to H^p(\mathcal{C}; G) = \mathrm{Ext}^p(A, G)$ is well-defined and functorial property is obvious.

2. (Hom-Ext sequence)
(1) Given a short exact sequence

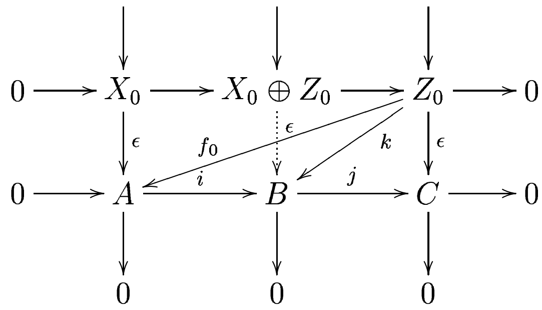
$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0,$$

want a long exact sequence (functorial) associated to it.
idea : Find free resolutions so that the diagram,

$$\begin{array}{ccccccccc} 0 \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow 0 \end{array}$$

commutes and apply snake lemma.

Start with X and Z, and then find Y. An obvious candidate is $Y = X \oplus Z$ with $\partial \oplus \partial$. Need to define ϵ at the final stage.



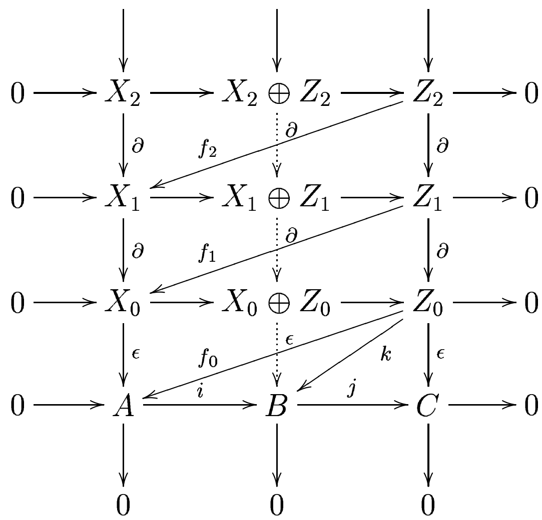
Since Z_0 is free, there exist $k : Z_0 \to B$ such that $jk = \epsilon$.
For any $f_0 : Z_0 \to A$, let

$$\epsilon(x, z) = i\epsilon(x) + f_0(z) + k(z).$$

(Write f_0 for if_0.)
Then this is the most general form of ϵ such that the above diagram commutes. We will try to find f_0 such that the middle vertical sequence is exact.

$$\epsilon(\partial x, \partial z) = i\epsilon(\partial x) + f_0(\partial z) + k(\partial z)$$

Since $jk(\partial z) = \epsilon(\partial z) = 0$, $k(\partial z) \in i(A)$. So need $f_0 = -k$ on $\partial Z_i = ker\epsilon$.
But in general, it is not possible to find such f_0. And hence we try to change boundary operator $\partial \oplus \partial$ in one higher step.



The most general form of ∂ such that the diagram commutes will be $\begin{pmatrix} \partial & f_1 \\ 0 & \partial \end{pmatrix}$ with $f_1 : Z_1 \to X_0$. Now put $f_0 = 0$ and

$$\begin{aligned} \epsilon(\partial(x,z)) &= \epsilon(\partial x + f_1 z, \partial z) \\ &= i\epsilon(\partial x + f_1 z) + k(\partial z) \\ &= i\epsilon(f_1 z) + k(\partial z) \end{aligned}$$

So we need to find f_1 such that $i\epsilon(f_1 z) = -k(\partial z)$. But this is possible because Z_1 is free and $jk(\partial z) = 0$.

Similarly given $\begin{pmatrix} \partial & f_1 \\ 0 & \partial \end{pmatrix}$, choose $f_2 : Z_2 \to X_1$
such that $0 = \partial^2 = \begin{pmatrix} \partial & f_2 \\ 0 & \partial \end{pmatrix} \begin{pmatrix} \partial & f_1 \\ 0 & \partial \end{pmatrix} = \begin{pmatrix} \partial^2 & \partial f_1 + f_2 \partial \\ 0 & \partial^2 \end{pmatrix}$.
This can be solved since Z is free and X is acyclic by the argument of comparison theorem. Inductively find $f_3, f_4, \cdots f_{p+1} : Z_{p+1} \to X_p$ to construct a chain complex Y. Since $H(X) = H(Z) = 0$, $H(Y) = 0$ by snake lemma, and hence we have found the desired resolutions.

Remark. What we really obtained here is a mapping cone $Y_p = X_p \oplus Z_p$ with respect to a chain map

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{-\partial} & Z_2 & \xrightarrow{-\partial} & Z_1 & \xrightarrow{-\partial} & Z_0 & \xrightarrow{-\epsilon} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle k} & & \Vert{\scriptstyle -id} & & \\ \cdots & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & X_0 & \xrightarrow{i \circ \epsilon} & B & \xrightarrow{j} & C & \longrightarrow & 0 \end{array}$$

(Check. Exercise)

(2) **Theorem.** If

$$0 \to A \to B \to C \to 0$$

is a short exact sequence, then there exists a natural(functorial) long exact sequence,

$$\begin{aligned} 0 &\to \operatorname{Hom}(C,G) \to \operatorname{Hom}(B,G) \to \operatorname{Hom}(A,G) \\ &\to \operatorname{Ext}^1(C,G) \to \operatorname{Ext}^1(B,G) \to \operatorname{Ext}^1(A,G) \to \operatorname{Ext}^2(C,G) \to \cdots \end{aligned}$$

Proof By (1) there exist free resolutions such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Taking $\mathrm{Hom}(-, G)$, since Z is free, we get a short exact sequence

$$0 \longleftarrow X^* \longleftarrow Y^* \longleftarrow Z^* \longleftarrow 0.$$

Applying snake lemma, we get a desired long exact sequence.

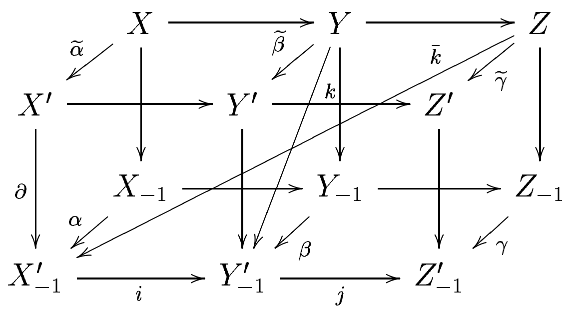
Functoriality follows from the following general consideration.
Suppose we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array} \quad \text{with resolutions} \quad \begin{array}{c} X, Y, Z \\ \\ X', Y', Z' \end{array}$$

Since the snake lemma is functorial, it is enough to show that there exist liftings such that the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\widetilde{\alpha}} & & \downarrow{\scriptstyle\widetilde{\beta}} & & \downarrow{\scriptstyle\widetilde{\gamma}} & & \\ 0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & 0 \end{array}$$

Consider the diagram.



Problem : Given two liftings $\widetilde{\alpha}, \widetilde{\gamma}$, find a lifting $\widetilde{\beta}$ inductively such that top squares commute.
First, we can always find $\widetilde{\beta} = \begin{pmatrix} \widetilde{\alpha} & f \\ 0 & \widetilde{\gamma} \end{pmatrix}$ such that top square commutes. But this may not be a lifting of β and modify $\widetilde{\beta}$ using f.
Consider $k = \partial\widetilde{\beta} - \beta\partial$. Then

$$jk = j(\partial\widetilde{\beta} - \beta\partial) = 0, \quad ki = (\partial\widetilde{\beta} - \beta\partial)i = 0$$

So, k induces $\bar{k} : Z \to X'_{-1}$ such that $i\bar{k}j = k$.[1]

[1] $jk = 0$ implies there exists $k' : Y \to X'_{-1}$ such that $ik' = k$. And since $ik'i = ki = 0$ and i is injective, $k'i = 0$. So, k' induces $\bar{k}$.

By induction hypothesis,

$$\partial(\partial\widetilde{\beta} - \beta\partial) = 0$$

So $\partial\bar{k} = 0$.[2]
Since X' is acyclic and Z is free, there exists $f : Z \to X'$ such that $\bar{k} = \partial f$.
Now "$\widetilde{\beta}$" $= \widetilde{\beta} - ifj$ is the desired lifting of β and note that "$\widetilde{\beta}$" still commutes the top squares.
Therefore we get commutative liftings of α, β, γ and the functoriality follows from the functoriality of snake lemma. □

3. (Why is the name Ext?)
Let A be an abelian group or R-module with R: P.I.D. (so that $\mathrm{Ext}^p = 0$ if $p \geq 2$).
Show $\mathrm{Ext}^1(A, B) \cong \mathrm{Ext}(A, B) := \{0 \to B \to E \to A \to 0\}/\sim$,
where $\{0 \to B \to E \to A \to 0\} \sim \{0 \to B \to E' \to A \to 0\}$ if and only if

$$\begin{array}{ccccccccc} 0 & \to & B & \to & E & \to & A & \to & 0 \\ & & \downarrow{=} & & \cong\downarrow\phi & & \downarrow{=} & & \\ 0 & \to & B & \to & E' & \to & A & \to & 0 \end{array}.$$

The existence of

$$0 \to R \xrightarrow{i} F \xrightarrow{\epsilon} A \to 0$$

implies the following long exact sequence

$$0 \to \mathrm{Hom}(A, B) \xrightarrow{\widetilde{\epsilon}} \mathrm{Hom}(F, B) \xrightarrow{\widetilde{i}} \mathrm{Hom}(R, B) \to \mathrm{Ext}^1(A, B) \to 0$$

Hence $\mathrm{Ext}^1(A, B) \cong \mathrm{Hom}(R, B)/\widetilde{i}(\mathrm{Hom}(F, B))(= \mathrm{coker}\ \widetilde{i})$
Given an extension

$$0 \to B \to E \to A \to 0,$$

Consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & F & \longrightarrow & A & \longrightarrow & 0 \\ & & \exists\alpha\downarrow & & \exists\beta\downarrow & & =\downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \uparrow{=} & \exists\alpha' & k\uparrow\cong & \exists\beta' & =\uparrow & & \\ 0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

free

acyclic

If $E' \overset{k}{\sim} E$, then $k \circ \beta', \alpha'$ are liftings of id.. By the comparison theorem $\alpha \simeq \alpha'$, i.e., $\exists D : F \to B$ such that $Di (= \widetilde{i}(D)) = \alpha' - \alpha$.

[2] Since $\partial k = 0$, $\partial i\bar{k}j = i\partial\bar{k}j = 0$. So $\partial\bar{k} = 0$ because i is injective and j is surjective.

$$\therefore [E] \stackrel{\Phi}{\longmapsto} [\alpha] \in \operatorname{coker} \widetilde{i},\ \forall E \in \operatorname{Ext}(A,B)$$

Show this correspondence is bijective :
Given $[\alpha]$, want an extension E such that

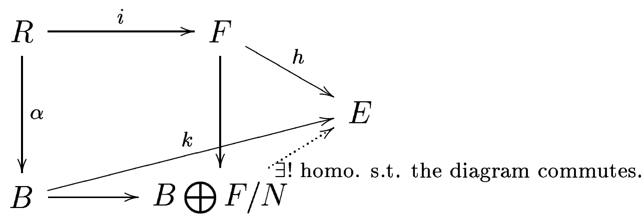
$$\begin{array}{ccccccccc} 0 & \rightarrow & R & \xrightarrow{i} & F & \xrightarrow{\epsilon} & A & \rightarrow & 0 \\ & & \downarrow \alpha & & \vdots \beta & & \downarrow = & & \\ 0 & \rightarrow & B & \underset{p}{\dashrightarrow} & E & \underset{q}{\dashrightarrow} & A & \rightarrow & 0 \end{array}$$

Use "push-out" of α and i to get E and β
Put $E = B \bigoplus F/N$ where $N = \{(-\alpha(r), i(r)) | r \in R\}$. (this forces $\alpha(r) = i(r)$ in E) and β, p, q are obvious maps.
Show if $\alpha \sim \alpha'$, then $E \sim E'$:
If $\alpha \sim \alpha'$, then $\alpha' = \alpha + Di$ where $D : F \to B$. Let $E' = B \bigoplus F/N', N' = \{(-\alpha'(r), i(r) | r \in R\}$(of course, $-\alpha'(r) = (-\alpha - Di)(r))$. Then

$$\begin{array}{ccccccccc} 0 & \rightarrow & N & \rightarrow & B \bigoplus F & \rightarrow & E & \rightarrow & 0 \\ & & \cong\downarrow \gamma| & & \cong\downarrow \gamma & & \vdots \bar{\gamma} & & \\ 0 & \rightarrow & N' & \rightarrow & B \bigoplus F & \rightarrow & E' & \rightarrow & 0 \end{array} \qquad \text{commutes}$$

if we define $\gamma = \begin{pmatrix} 1 & -D \\ 0 & 1 \end{pmatrix}$ and γ induces $\bar{\gamma} : E \to E'$ and $\bar{\gamma}$ is an isomorphism (by 5-lemma). Therefore $E \sim E'$.
This construction is clearly the inverse of Φ(check: Exercise).

Note The "push-out" has a universal property, i.e., when with $h \circ i = k \circ \alpha$,



Remark Similar interpretation of $\operatorname{Ext}^p(A,B)$
(See MacLane or Hilton and Stambach)

4. Universal coefficient Theorem

Let $\mathcal{C}$ be a free chain complex and G an abelian group or R-module with R, a P.I.D. Then there exists a natural short exact sequence for all p

$$0 \to \operatorname{Ext}(H_{p-1}(\mathcal{C}), G) \to H^p(\mathcal{C}; G) \to \operatorname{Hom}(H_p(\mathcal{C}), G) \to 0$$

which splits (but not naturally).

증명

$$\cdots \to C_p \xrightarrow{\partial} C_{p-1} \xrightarrow{\partial} \cdots \qquad : \mathcal{C}$$

Consider

$$0 \to B_p \xrightarrow{i} Z_p \xrightarrow{p} H_p \longrightarrow 0 \qquad \text{and}$$

$$0 \to Z_p \underset{j}{\to} C_p \underset{\partial}{\to} B_{p-1} \to 0$$

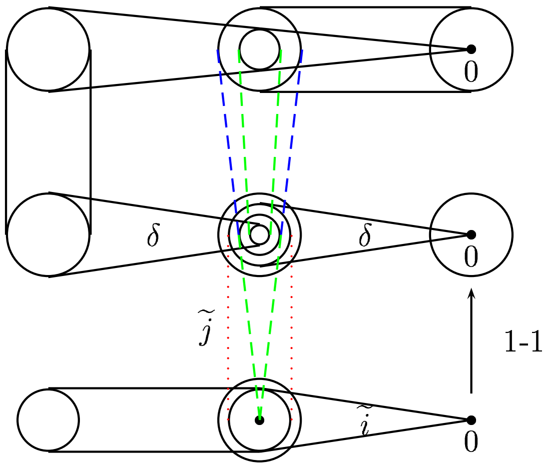
Notation $A^{'} = \operatorname{Hom}(A, G)$ and $\operatorname{Ext}_p = \operatorname{Ext}(H_p, G)$
By applying Hom-functor, we obtain

$$0 \longrightarrow H_p^{'} \longrightarrow Z_p^{'} \to B_p^{'} \to \operatorname{Ext}_p \to 0$$

$$0 \to B_{p-1}^{'} \xrightarrow{\delta} C_p^{'} \to Z_p^{'} \longrightarrow 0$$

Note that since Z_p is free, $\operatorname{Ext}(Z_p, G) = 0$. Now consider the following diagram and figure.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & 0 \\
 & & \uparrow & & \downarrow & & & & \uparrow \\
0 \to H_{p-1}^{'} & \to & Z_{p-1}^{'} & \longrightarrow & B_{p-1}^{'} & \to \operatorname{Ext}_{p-1} \to 0 & & & Z_{p+1}^{'} \\
 & & \uparrow & & \downarrow & & & & \uparrow \\
\cdots & \longrightarrow & C_{p-1}^{'} & \xrightarrow{\delta} & C_p^{'} & & \xrightarrow{\delta} & & C_{p+1}^{'} \longrightarrow \cdots \\
 & & \uparrow & & \tilde{j}\downarrow & & & & \uparrow \\
 & & B_{p-2}^{'} & 0 \to H_p^{'} \longrightarrow & Z_p^{'} & & \underset{\tilde{i}}{\longrightarrow} & & B_p^{'} \longrightarrow \operatorname{Ext}_p \to 0 \\
 & & \uparrow & & \downarrow & & & & \uparrow \\
 & & 0 & & 0 & & & & 0
\end{array}$$



$\widetilde{j}(\ker\delta) = \ker\widetilde{i} \cong H_p'$

Naturality follows from the naturality of Hom − Ext sequence and of the above construction in the proof.

□

따름정리 1 *Let (X, A) be a pair of spaces. Then there exists natural short exact sequence which splits*

$$0 \rightarrow Ext(H_{p-1}(X,A), G) \rightarrow H^p(X, A; G) \rightarrow Hom(H_p(X,A), G) \rightarrow 0$$

(or as a direct sum)

Note If H_{p-1} is free, then $H^p(\mathcal{C}; G) \cong \mathrm{Hom}(H_p(\mathcal{C}), G)$.(e.g. R is a field)

숙제 17 Compute $H^*(\text{closed surfaces}), H^*(\mathbb{R}P^n)$ and $H^*(\mathbb{C}P^n)$(See 5.)

5. Computation of Ext
Recall $\mathrm{Ext}(\text{free}, G) = 0$

정리 2 *(1)* $Ext(\bigoplus A_\alpha, G) \cong \prod_\alpha Ext(A_\alpha, G)$
$Ext(A, \prod G_\alpha) \cong \prod_\alpha Ext(A, G_\alpha)$
(2) $Ext(\mathbb{Z}/n, G) \cong G/nG$

증명 (1)

$$0 \to R_\alpha \to F_\alpha \to A_\alpha \to 0 \qquad \text{free resolution}$$

$\Rightarrow$

$$0 \to \bigoplus R_\alpha \to \bigoplus F_\alpha \to \bigoplus A_\alpha \to 0 \qquad \text{free resolution}$$

$\Rightarrow$

$$0 \longrightarrow \text{Hom}(A_\alpha, G) \longrightarrow \text{Hom}(F_\alpha, G) \longrightarrow \text{Hom}(R_\alpha, G) \longrightarrow \text{Ext}(A_\alpha, G) \longrightarrow 0 \qquad \clubsuit$$

$$0 \to \text{Hom}(\bigoplus A_\alpha, G) \to \text{Hom}(\bigoplus F_\alpha, G) \to \text{Hom}(\bigoplus R_\alpha, G) \to \text{Ext}(\bigoplus A_\alpha, G) \to 0 \qquad \clubsuit\clubsuit$$

Note that $\text{Hom}(\bigoplus A_\alpha, G) = \prod \text{Hom}(A_\alpha, G)$, etc. Now apply $\prod_\alpha$ to $\clubsuit$ and compare with $\clubsuit\clubsuit$ to get the result.(5-lemma)
For 2nd isomorphism, similar argument using $\text{Hom}(A, \prod G_\alpha) = \prod \text{Hom}(A, G_\alpha)$

(2)

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n \to 0 \qquad \text{free resolution}$$

$\Rightarrow$

$$0 \to \text{Hom}(\mathbb{Z}/n, G) \to \text{Hom}(\mathbb{Z}, G)(= G) \xrightarrow{\times n} \text{Hom}(\mathbb{Z}, G)(= G) \to \text{Ext}(\mathbb{Z}/n, G) \to 0$$

Hence $\text{Ext}(\mathbb{Z}/n, G) \cong \text{coker}(\times n) = G/nG$ and $\text{Hom}(\mathbb{Z}/n, G) = \ker(\times n)$.

□

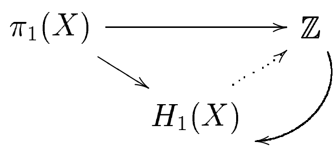
숙제 18
$\text{Hom}(\mathbb{Z}/n, \mathbb{Z}/m) = \text{Ext}(\mathbb{Z}/n, \mathbb{Z}, m) = \mathbb{Z}/d$, where $d = (m, n)$.

6. Let X be a CW-complex.

$$\begin{array}{ccc} [X, S^1] & \xrightarrow{\cong} & H^1(X, \mathbb{Z}) \\ {\scriptstyle (a)}\downarrow\cong & & {\scriptstyle (c)}\downarrow\cong \\ \text{Hom}(\pi_1 X, \pi_1 S^1) & \underset{(b)}{\to} & \text{Hom}(H_1(X), \mathbb{Z}) \end{array}$$

where $[X, Y] = \text{Maps}(X, Y)/\simeq = \{[f] : \text{homotopy class} | f : X \to Y\}$.
(1) (a)를 보이기 위해 $[X, S^1] \cong [X, S^1]_* \cong \text{Hom}(\pi_1 X, \pi_1 S^1)$ 임을 증명하자. 여기서 $[X, S^1]_*$는 X와 S^1에 base point 가 정해져 있는 경우를 생각한 것이다. 그런 다음 $\text{Hom}(\pi_1 X, \pi_1 S^1) \cong [X, S^1]_*$을 보이기 위해서는 X에 있는 maximal tree 를 생각하고, 이 maximal tree 를 contract 시켜 base point로 잡은 후

$\pi_1 X$와 $\pi_1 S^1$사이의 주어진 map에 대해 대응하는 cellular map : $X \to S^1$을 잘 정의할 수 있다. ($\pi_1(X)$의 presentation을 이용하고, $\pi_i(S^1) = 0, i \geq 2$이라는 사실을 이용하여)

(2) 

(b)를 보이기 위해서는 우선 $\pi_1(X)$를 abelize 하면 $H_1(X)$이 된다는 것에 주목하자. 따라서 우리는 왼쪽의 diagram을 commute시키는 map을 찾으면 된다.

(3) (c)는 universal coefficient thoerem에 의해서 자명하다.

숙제 19(Prove in detail)

Fact $[X, K(\pi, n)] \cong H^n(X; \pi)$

여기서 $X \in K(\pi, n) \Leftrightarrow X$가 $\pi_n(X) = \pi$이고 $k \neq n$ 일때 $\pi_k(X) = 0$인 space이다. 예를 들면, $S^1 \in K(\mathbb{Z}, 1)$이다.